

Fourth order Weyl Gravity

Éanna É. Flanagan*

Laboratory for Elementary Particle Physics, Cornell University, Ithaca, NY 14853, USA.

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The fourth order Weyl gravity theory of Mannheim and Kazanas is based on replacing the Einstein-Hilbert action with the square of the Weyl tensor, and on modifying the matter action of the standard model of particle physics to make it conformally invariant. This theory has been suggested as a model of both dark matter and dark energy. We argue that the conformal invariance is not a fundamental property of the theory, and instead is an artifact of the choice of variables used in its description. We deduce that in the limit of weak fields and slow motions the theory does not agree with the predictions of general relativity, and is therefore ruled out by Solar System observations.

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I. INTRODUCTION

In the fourth order Weyl theory of gravity [1, 2, 3, 4, 5, 6, 7, 8, 9], the Einstein-Hilbert action is replaced by a term proportional to the square of the Weyl tensor, and the action of the standard model of particle physics is modified to make it be conformally invariant. For simplicity and following Ref. [9] we work here with a subset of the standard model consisting of a Dirac fermion field ψ , a gauge field A_α and a real scalar field S which plays the role of the Higgs field. The action of the theory is a functional of these fields and of a metric $g_{\alpha\beta}$:

$$S[g_{\alpha\beta}, S, \psi, A_\alpha] = \int d^4x \sqrt{-g} \left\{ -\alpha_g C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} - \frac{1}{2} \nabla_\alpha S \nabla^\alpha S - \frac{1}{12} S^2 R - \lambda S^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \gamma^\mu \nabla_\mu \psi - \frac{i}{2} \overline{\nabla_\mu \psi} \gamma^\mu \psi + e \bar{\psi} \gamma^\mu A_\mu \psi - h S \bar{\psi} \psi \right\}. \quad (1)$$

Here α_g , h and λ are dimensionless parameters, we use natural units with $\hbar = c = 1$, and we use the sign convention $(+, +, +)$ in the notation of Ref. [10]¹.

The motivations for the action (1) are as follows [9]. First, it is invariant under the conformal transformations

$$g_{\alpha\beta} \rightarrow e^{2\sigma} g_{\alpha\beta} \quad (2a)$$

$$S \rightarrow e^{-\sigma} S \quad (2b)$$

$$\psi \rightarrow e^{-3\sigma/2} \psi \quad (2c)$$

$$A_\alpha \rightarrow A_\alpha, \quad (2d)$$

where $\sigma = \sigma(x)$ is arbitrary. This exact symmetry prevents the appearance of a cosmological constant. Second,

it was argued that the term coupling the Ricci scalar to the scalar field S can drive a gravity-mediated spontaneous symmetry breaking: namely, in the presence of a background value of R , the minimum energy state of S will occur at a nonzero value of S and will thereby give mass to the fermion field. In Refs. [1, 2, 5, 6] it was argued that the theory (1) agrees with observations of Newtonian gravity in the Solar System, and in addition predicts a linearly growing term in the Newtonian potential that could explain galactic rotation curves without the need for dark matter. Refs. [7, 8, 9] argue that Weyl gravity yields a viable model of the acceleration of the Universe, removing the need for dark energy. Further studies of the theory can be found in Refs. [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

In this paper we rewrite the theory (1) in a new set of variables that allows a simpler computation of its predictions. We also show that the theory does not reproduce Solar System observations in the limit of weak fields and slow motions, which rules out the theory.

II. REFORMULATION OF THEORY

We specialize at first to the sector of the theory where

$$S(x) > 0 \quad (3)$$

everywhere². We define the new variables

$$\hat{g}_{\alpha\beta} = \frac{S^2}{m_0^2} g_{\alpha\beta} \quad (4a)$$

$$\hat{S} = S \quad (4b)$$

$$\hat{\psi} = \frac{m_0^{3/2}}{S^{3/2}} \psi \quad (4c)$$

$$\hat{A}_\alpha = A_\alpha, \quad (4d)$$

*Electronic address: eef3@cornell.edu

¹ Our sign conventions differ from those used by Mannheim in Ref. [9]

² It follows from our analysis below that if S is positive on an initial data surface, then it is positive throughout spacetime.

where m_0 is an arbitrary but fixed positive parameter with dimensions of mass. All of these variables, except \hat{S} , are conformal invariants. The action in terms of the new variables is³

$$S[\hat{g}_{\alpha\beta}, \hat{S}, \hat{\psi}, \hat{A}_\alpha] = \int d^4x \sqrt{-\hat{g}} \left\{ -\alpha_g \hat{C}_{\alpha\beta\gamma\delta} \hat{C}^{\alpha\beta\gamma\delta} - \frac{1}{12} m_0^2 \hat{R} - \lambda m_0^4 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{i}{2} \bar{\hat{\psi}} \hat{\gamma}^\mu \hat{\nabla}_\mu \hat{\psi} - \frac{i}{2} \overline{\hat{\nabla}_\mu \hat{\psi}} \hat{\gamma}^\mu \hat{\psi} + e \bar{\hat{\psi}} \hat{\gamma}^\mu \hat{A}_\mu \hat{\psi} - h m_0 \bar{\hat{\psi}} \hat{\psi} \right\}. \quad (5)$$

In this new representation, the only field which transforms under the conformal symmetry (2) is \hat{S} . However, the action (5) is independent of \hat{S} . Thus there are two uncoupled sectors of the theory (1) with the constraint (3): a trivial sector containing \hat{S} on which the symmetry acts, and which contains no dynamics; and the remaining sector containing the fields $\hat{g}_{\alpha\beta}$, \hat{A}_α and $\hat{\psi}$, which does not possess a conformal symmetry. For the remainder of this paper, we will drop the field \hat{S} and consider only the dynamical sector of the theory.

In a similar manner, one can start from the action for general relativity coupled to the standard model of particle physics, perform the above operations in reverse, and obtain an equivalent action with one extra scalar field which has an exact conformal symmetry. It follows that the conformal symmetry of the theory (1) is not a fundamental or defining property of the theory, and is instead an artifact of the choice of variables used to describe the theory.

The transition from the action (1) to the action (5) can also be thought of as a gauge fixing [22, 23]. The conformal symmetry (2) is analogous to a gauge freedom, and we are free to analyze the theory in the gauge $S(x) = m_0$, which leads to the action (5).

In the action (5), the parameter m_0 can be chosen arbitrarily. This arbitrariness is the freedom of choice of units of mass. Only the ratios of the three mass parameters which appear in the action are measurable. These three mass parameters are the Planck mass (the coefficient of Ricci scalar), the cosmological constant term, and the mass term for the fermion field. If we define $m_p^2 = m_0^2/6$, $\Lambda = \lambda m_0^4$, and $m_e = h m_0$, then the action

can be written in the more familiar looking form

$$S[\hat{g}_{\alpha\beta}, \hat{S}, \hat{\psi}, \hat{A}_\alpha] = \int d^4x \sqrt{-\hat{g}} \left\{ -\alpha_g \hat{C}_{\alpha\beta\gamma\delta} \hat{C}^{\alpha\beta\gamma\delta} - \frac{1}{2} m_p^2 \hat{R} - \Lambda - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{i}{2} \bar{\hat{\psi}} \hat{\gamma}^\mu \hat{\nabla}_\mu \hat{\psi} - \frac{i}{2} \overline{\hat{\nabla}_\mu \hat{\psi}} \hat{\gamma}^\mu \hat{\psi} + e \bar{\hat{\psi}} \hat{\gamma}^\mu \hat{A}_\mu \hat{\psi} - m_e \bar{\hat{\psi}} \hat{\psi} \right\}. \quad (6)$$

This is the standard action for a fermion field of mass m_e and charge e coupled to a gauge field and coupled to general relativity, except for three modifications to the gravitational part of the action: (i) the addition of the cosmological constant term; (ii) the sign of the Ricci term is flipped, and (iii) the Weyl squared term is added. Note also that the value of the Planck mass parameter m_p can be different from its conventional value of $\sim 10^{19}$ GeV; below we will consider all possible values of the parameters α_g and m_p .

We next return to the original action (1), and consider the sector of the theory where $S(x) < 0$ everywhere. A similar analysis shows that this sector is also described by an action of the form (6), but with the sign of the fermion mass term flipped. This can be compensated for by redefining $\psi \rightarrow \gamma^5 \psi$. Thus the $S < 0$ sector behaves the same way as the $S > 0$ sector. We will confine attention to the $S > 0$ sector.

III. WEAK FIELD LIMIT

Consider now the predictions of the theory (6) in the limit of weak fields and slow motions. A key point is that the physical metric measured by experiments⁴ is the metric $\hat{g}_{\alpha\beta}$, and not the metric $g_{\alpha\beta}$ that appeared in the original action (1). This follows from the form of the action (6), which has a standard form that implies that objects constructed from the fermion and gauge fields will fall on geodesics of $\hat{g}_{\alpha\beta}$ ⁵. Equivalently, in terms of the original variables $g_{\alpha\beta}$ and S , all freely falling objects are subject to an acceleration proportional to the gradient of S , as argued by Wood [23].

It is straightforward to show that the theory (6) does not admit a regime in which its predictions agree with the weak-field slow-motion limit of general relativity, for any choice of values of the parameters α_g and m_p , which

³ Here $\hat{\gamma}^\mu$ are the Dirac matrices associated with the metric $\hat{g}_{\mu\nu}$ that satisfy $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = -2\hat{g}^{\mu\nu}$.

⁴ Here it is assumed that the units for length and time are defined using non-gravitational physics. Systems of units in common use such as SI units satisfy this requirement.

⁵ It is immediately clear that $g_{\alpha\beta}$ cannot be the physical metric, since the theory cannot predict $g_{\alpha\beta}$ uniquely, only $g_{\alpha\beta}$ up to conformal transformations. By contrast, the metric $\hat{g}_{\alpha\beta}$ can be predicted uniquely.

implies that the theory is ruled out by Solar System observations. Substituting the ansatz

$$\hat{g}_{ab}dx^a dx^b = -[1 + 2\Phi(\mathbf{x})]dt^2 + [1 - 2\Psi(\mathbf{x})]\delta_{ij}dx^i dx^j \quad (7a)$$

$$T_{ab}dx^a dx^b = \rho(\mathbf{x})dt^2 \quad (7b)$$

into the linearized equation of motion obtained from the action (6) yields the solution

$$\Phi = \frac{4}{3}\bar{\Phi} + \frac{1}{3}\Phi_N \quad (8a)$$

$$\Psi = \frac{2}{3}\bar{\Phi} - \frac{1}{3}\Phi_N. \quad (8b)$$

Here Φ_N is the usual Newtonian potential which satisfies

$$2m_p^2 \nabla^2 \Phi_N = \rho, \quad (9)$$

where ρ is the mass density, and $\bar{\Phi}$ is a potential which satisfies the fourth order equation

$$8\alpha_g \nabla^2 \nabla^2 \bar{\Phi} - 2m_p^2 \nabla^2 \bar{\Phi} = \rho. \quad (10)$$

We have neglected the cosmological constant term whose influence will be negligible on Solar System scales and smaller scales.

We now consider an isolated source of mass $\sim M$ and size $\sim L$. Some useful information can be obtained from dimensional analysis. In a general system of units with $\hbar \neq 1$, the action (6) contains two independent dimensionful parameters, the mass scale $\sqrt{\alpha_g}m_p$ and the lengthscale $\sqrt{\alpha_g}/m_p$. There are therefore two dimensionless parameters characterizing the source, namely $Lm_p/\sqrt{\alpha_g}$ and $M/(\sqrt{\alpha_g}m_p)$. There are three different regimes in this two-dimensional parameter space in which the theory exhibits different types of behavior (see Fig. 1):

$$\text{regime 1 : } \frac{Lm_p}{\sqrt{\alpha_g}} \gg \frac{M}{\sqrt{\alpha_g}m_p}, \quad \frac{Lm_p}{\sqrt{\alpha_g}} \gg 1 \quad (11a)$$

$$\text{regime 2 : } \frac{Lm_p}{\sqrt{\alpha_g}} \gg \frac{M}{\sqrt{\alpha_g}m_p}, \quad \frac{Lm_p}{\sqrt{\alpha_g}} \ll 1 \quad (11b)$$

$$\text{regime 3 : } \frac{Lm_p}{\sqrt{\alpha_g}} \lesssim \frac{M}{\sqrt{\alpha_g}m_p}. \quad (11c)$$

We now discuss these various regimes in turn.

A. Regime 1

Consider the ratio between the first and second terms in Eq. (10), evaluated in the vicinity of the source at $r \sim L$. This ratio is $\sim \alpha_g/(m_p^2 L^2)$, which is small compared to unity by Eq. (11a). Consequently the fourth order derivative term gives only a small correction, and it follows that $\bar{\Phi} \approx -\Phi_N$ and so also $\Phi = \Psi = -\Phi_N$ from Eqs. (8). Thus Newton's law of gravitation is recovered but with the sign flipped. The resulting repulsive gravitational force disagrees with observations. This conclusion applies in particular for the conventional values of the parameters, namely $\alpha_g \sim 1$ and $m_p \sim 10^{19}$ GeV.

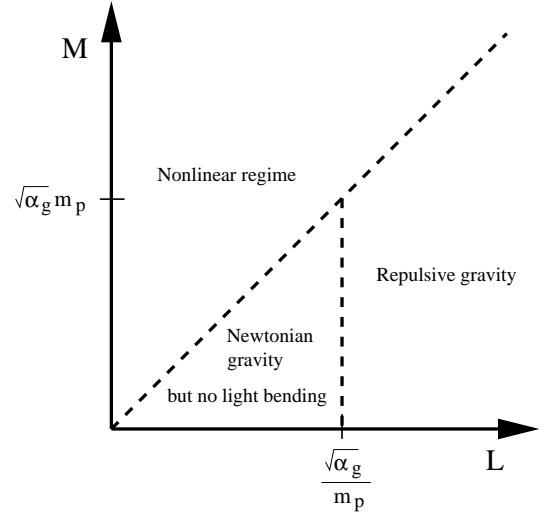


FIG. 1: The various regimes for fourth order Weyl gravity for a source of mass $\sim M$ and size $\sim L$.

B. Regime 2

The exact solution to Eq. (10) is

$$\bar{\Phi}(\mathbf{x}) = -\frac{1}{4\pi} \int d^3y \frac{\chi(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}, \quad (12)$$

where⁶

$$\chi(\mathbf{y}) = -\frac{1}{32\pi\alpha_g} \int d^3z \frac{e^{-\frac{1}{\sqrt{2}\alpha_g}m_p|\mathbf{y}-\mathbf{z}|}}{|\mathbf{y} - \mathbf{z}|} \rho(\mathbf{z}). \quad (13)$$

Using this solution we obtain the order of magnitude estimates $\chi(r) \sim M/(\alpha_g L)$ for $r \sim L$, $\chi(r) \sim M/(\alpha_g r)$ for $L \ll r \ll \sqrt{\alpha_g}/m_p$, while χ falls off exponentially for $r \gtrsim \sqrt{\alpha_g}/m_p$. In spherical symmetry the gradient of the field $\bar{\Phi}$ is

$$\frac{\partial \bar{\Phi}}{\partial r} \sim \frac{1}{r^2} \int_0^r dr' (r')^2 \psi(r') \quad (14)$$

which yields the estimate $\bar{\Phi}_{,r} \sim M/\alpha_g$ for $r \sim L$, $\bar{\Phi}_{,r} \sim M/\alpha_g$ for $L \ll r \ll \sqrt{\alpha_g}/m_p$, and $\bar{\Phi}_{,r} \sim M/(m_p^2 r^2)$ for $r \gg \sqrt{\alpha_g}/m_p$. Therefore for $r \ll \sqrt{\alpha_g}/m_p$ the acceleration produced by the potential $\bar{\Phi}$ is smaller than the acceleration $\sim M/(m_p^2 r^2)$ produced by the Newtonian potential term Φ_N in the expression (8a) for Φ by a factor of $\sim m_p^2 r^2/\alpha_g \ll 1$. Therefore to a good approximation the solution in this regime is given by⁷ Eqs. (8)

⁶ If α_g is negative rather than positive as assumed here, the exponential in Eq. (13) is replaced by a cosine. This replacement does not qualitatively change the subsequent discussion.

⁷ Up to an overall constant term in $\bar{\Phi}$ which can be eliminated by a gauge transformation of the form $x^i \rightarrow \alpha x^i$, $t \rightarrow \alpha^{-2}t$ and which is not locally measurable.

with the $\bar{\Phi}$ terms dropped:

$$\Phi = \frac{1}{3}\Phi_N, \quad \Psi = -\frac{1}{3}\Phi_N. \quad (15)$$

Since the motion of massive particles is governed by the potential Φ , we see that Newtonian gravity is recovered locally for massive particles with an effective Newton's constant $G_{\text{eff}} = 1/(24\pi m_p^2)$.

The problem which occurs in regime 2 is light bending. Since the metric given by Eqs. (7a) and (15) is conformally flat to a good approximation, there is no light bending. More precisely, the amount of light bending produced for a ray that grazes the source is smaller than the prediction of general relativity by a factor of $\sim L^2 m_p^2 / \alpha_g \ll 1$. Another way of describing this is in terms of the Eddington PPN parameter γ , defined by the metric expansion in spherical symmetry

$$ds^2 = - \left[1 - \frac{2\bar{M}}{r} + O(1/r^2) \right] dt^2 + \left[1 + \frac{2\gamma\bar{M}}{r} + O(1/r^2) \right] \delta_{ij} dx^i dx^j. \quad (16)$$

Comparing Eqs. (7a) and (15) with Eq. (16) yields

$$\gamma = -1 + O\left(\frac{L^2 m_p^2}{\alpha_g}\right). \quad (17)$$

Experimentally it is known that $\gamma = 1$ to a within a small fraction of a percent, in agreement with the prediction of general relativity. The deflection of a ray of light is proportional to $1 + \gamma$.

C. Regime 3

We first note that the linearized equations of motion are a good approximation in regimes 1 and 2. The poten-

tial $\Phi_N \sim M/(m_p^2 L)$ defined by Eq. (9) is small compared to unity by Eqs. (11a) and (11b). In regime 1 we have $\bar{\Phi} \approx -\Phi_N$ so $|\bar{\Phi}| \ll 1$. In regime 2, the largest value of $\bar{\Phi}$ is of order $(M/\alpha_g)(\sqrt{\alpha_g}/m_p) \sim M\sqrt{\alpha_g}/m_p$ which is small compared to unity by Eqs. (11b).

In regime 3 however, we have $|\Phi_N| \gtrsim 1$ from Eq. (11c), and so the linearized approximation breaks down. In this regime one must use the full nonlinear equations of the theory. However, it is clear that Newtonian phenomenology cannot be reproduced in this regime since the linear superposition principle will not apply.

Finally, a separate problem with the theory (6) is that it contains a ghost field, i.e. a field whose kinetic energy term has the wrong sign. This ghost field is a massless spin 2 field that is due to the negative Ricci term in the action (6) [22]. There is also a massive spin 2 field in the theory; this field is normally ghostlike [24] but here is not, due to the negative coefficient of the Ricci scalar. It is however tachyonic for $\alpha_g > 0$.

As this paper was being completed, we learned that similar arguments had been presented by Karel Van Acoleyen at a conference [25].

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